Options, American Style

Comparison of American Options and European Options

Background on Stocks

- On time domain [0, T], an asset (such as a stock) changes in value from S_0 to S_T
- At each period n, the value of S_n is binomial, since there are basically only two possible outcomes
  - $S_n > S_{n-1}$ (by a factor u with probability p)
  - $S_n < S_{n-1}$ (by a factor d with probability $1–p$).

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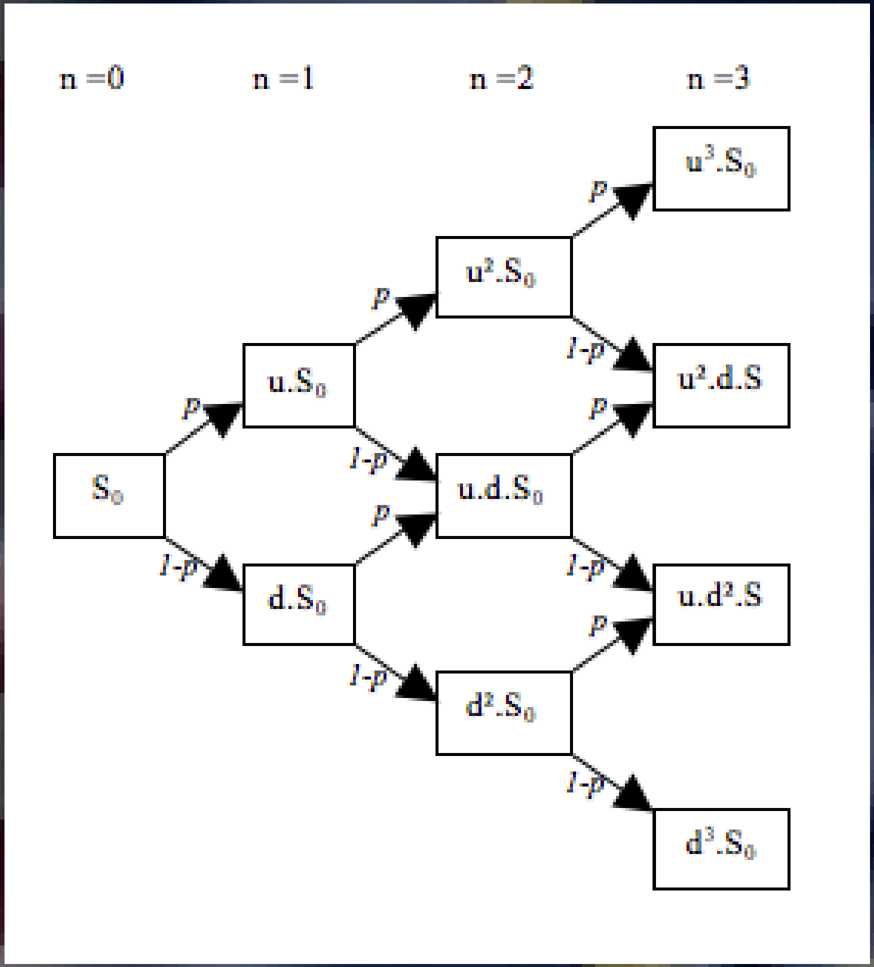
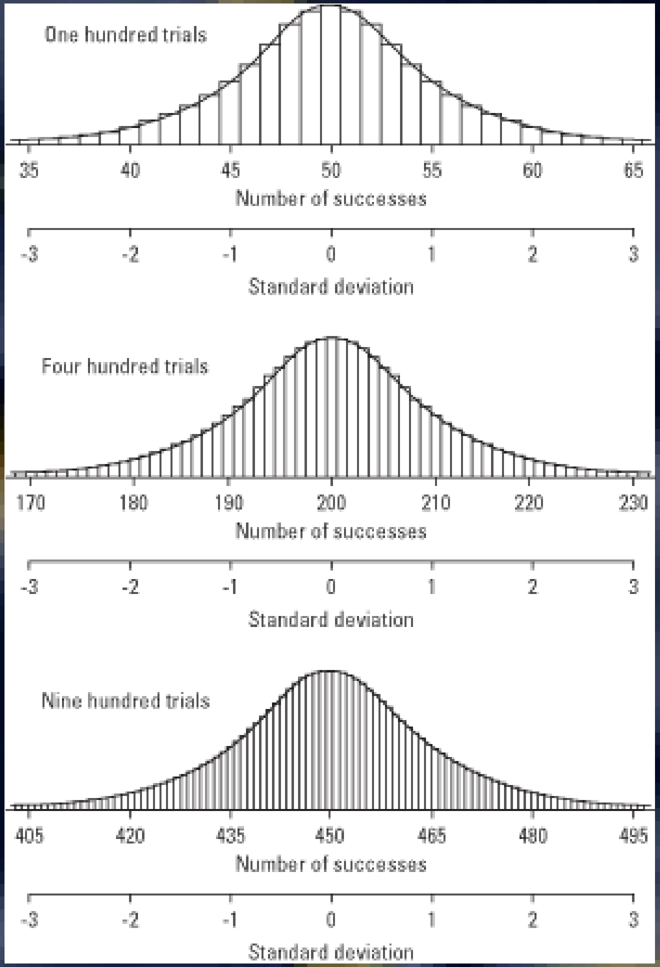
Background on Stocks

- Over multiple periods, this creates a binomial distribution with many trials



Background on Stocks

- As the number of trials increases, a binomial (discrete) distribution approaches a normal (continuous) distribution.



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Background on Stocks

- With a normal curve involved, then, this implies a mean that can be tied to an interest rate r on a risk-free asset (such as a bond), and a standard deviation σ that is equivalent to the volatility of the stock.

Background on Options

- An option is the ability to exercise (buy or sell) the stock at pre-fixed price K at some time t < T.
- "Call" versus "Put"
  - Call: Ability to buy at time t; hopefully $K < S_t$
  - Put: Ability to sell at time t; hopefully $K > S_t$
- Variations
  - Varying the strike price, e.g. forward start
  - Varying the strike time, e.g. shout

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Background on Options

- On time domain [0, t], the option itself has a value that changes from V_0 to V_t.
- At time t, V_t is the difference between spot price S_t and strike price K if exercised or zero if not, so $V_t^{©/Ⓟ} = \max\{\pm(S_t - K), 0\}$
  - Call option: $V_t = \max\{S_t - K, 0\}$
  - Put option: $V_t = \max\{K - S_t, 0\}$

Background on Options

- Problem: We know at time t, there are only two values for V_t. But the option must be priced at time 0, so how should V_0 be calculated?

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Background on Options

- Solution: The future value should be a function of the initial value and the interest rate, so V_0 is the inverse function of V_t.
  - For example, assuming continuous growth, the payoff $V_t = V_0 e^{rt}$, so $V_0 = e^{-rt} V_t$.

Background on Options

- Problem: We know the initial value of the option depends on its value at maturity. But that is an unknown quantity, so how should V_t be calculated?

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Background on Options

- <u>Solution</u>: V_t is uncertain, but it does have an expected value, so V_t can be replaced by $E(V_t)$.
  - For example, if at maturity there is an equal chance of exercising the option ($V_t = |S_t – K|$) or not doing so ($V_t = 0$), then $E(V_t) = ½|S_t – K|$
  - If not equal, then the probabilities must be weighted, i.e. $E(V_t) = p|S_t – K|$

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Comparison of Options

American Option

- Ability to exercise the option at any time on the interval [0,*t*]
- Traded on futures exchanges
- Most stock and equity options
- More likely to be auto-exercised

European Option

- Ability to exercise the option at fixed *t*
- Usually traded over-the-counter
- Most indexes, e.g. S&P500
- Can be auto-exercised

European Options

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

- The typical method of valuing European options is the Black-Scholes-Merton model.
- This is a PDE relating how fast V_t changes over time, how much V_t changes compared to S_t, and the parameters r and σ.

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European Options

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} =$$

- L.H.S. represents change of value over time...
  - negative because there is less time to exercise
- ...plus a term that reflects the gain in holding on to the option
  - presumably positive because $\sigma^2 S^2 > 0$ and a (hopefully) upward concavity of the value with respect to the stock

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European Options

$$= rV - rS\frac{\partial V}{\partial S}$$

- R.H.S. represents the return from a long position (buying it)…
  - depends only on the payoff at maturity
- …and a short position (selling it)
  - depends on the amount lost as a function of the positive slope of payoff against the underlying stock

European Options

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

- Thus, the Black-Scholes model says how a European option can be valued because the overall loss and gain with it (L.H.S.) offset each other to equal the return at the riskless rate (R.H.S.)

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European Options

- Solving the PDE yields $E(V_t^{©/Ⓟ}) = \pm(S_t N(\pm d_+) - K N(\pm d_-))$
  - N is the normal cdf
  - $d_\pm = (\ln(S_t/K) \pm ½\sigma^2 t)/(\sigma\sqrt{t})$
- Multiply by the discount factor e^{-rt} to get V_0
  - $V_0^{©} = e^{-rt}E(V_t^{©}) = S_0 N(d_+) - Ke^{-rt}N(d_-)$
  - $V_0^{Ⓟ} = e^{-rt}E(V_t^{Ⓟ}) = -S_0 N(-d_+) + Ke^{-rt}N(-d_-)$

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American Options

- There is no consensus as to how to price an American option
- There is also no optimal strategy on when to exercise the option, though some cases are obvious
  - Exercise a put if the asset files for bankruptcy
  - Exercise a put if the asset is high and traditionally holds value (e.g. gold)
  - Exercise a call if the asset is high and is about to pay a dividend that would lower its value too much to recover

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American Options

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \leq 0$$

- Black-Scholes can be modified to price an American option.
  - Given an option of each style with identical parameters, the American option should be worth more, as it is intrinsically more valuable to have more points at which to exercise it.

American Options

- Black's Approximation: Compute and take the higher of the following:
  - a European call where the present stock value is reduced by the present dividend value(s), i.e. $S_0' = S_0 - \sum_i^n (D_i e^{-rt_i})$
  - a European call where the present stock is reduced by all but the last dividend and set to expire on the day before that last dividend, i.e. $S_0' = S_0 - \sum_i^{n-1} (D_i e^{-rt_i})$ and $t = t_n$.

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American Options

- Binomial Options Pricing Model
  - Construct the binary price tree with $u = e^{\sigma\sqrt{t}}$ and $d = e^{-\sigma\sqrt{t}} = 1/u$. If there are U upticks and D downticks, this process can be sped up by noting that $S_n = S_0 u^U d^D = S_0 u^U (1/u)^D = S_0 u^{U-D}$
  - Find the value of the option at the terminal nodes, i.e. $\max\{\pm(S_n - K), 0\}$
  - Find the value of the option at all parent nodes. For each node, this means calculating a binomial value using the child nodes (weighted by their probabilities), applying the discount factor, then taking max{binomial value, exercise value}

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American Options

- Barone-Adesi & Whaley model: Adjust a European option with an early exercise premium

$$C(S) = c + A_2\left(\frac{S}{S^*}\right)^{q_2} \quad \text{if } S < S^*$$
$$S - X \quad \text{if } S \geq S^*$$

$$P(S) = p + A_1\left(\frac{S}{S^{**}}\right)^{q_1} \quad \text{if } S > S^*$$
$$X - S \quad \text{if } S \leq S^*$$

$$A_2 = \frac{S^*[1 - e^{(q-r)(T-t)}N(d_1(S^*))]}{q_2}$$

$$A_1 = -\frac{S^{**}[1 - e^{(q-r)(T-t)}N(-d_1(S^{**}))]}{q_1}$$

$$q_1 = \frac{1 - n - \sqrt{(n-1)^2 + 4k}}{2}; \quad q_2 = \frac{1 - n + \sqrt{(n-1)^2 + 4k}}{2}; \quad n = \frac{2(r-q)}{\sigma^2}; \quad k = \frac{2r}{\sigma^2(1 - e^{-r(T-t)})}$$

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