Crossing the Rubik's Cube

(When Julius forded the Rubicon, was that the first Caesar Shift?)

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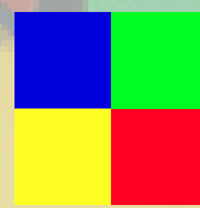
Rubik's… Square?

- Before tackling the problem in three dimensions, how about the 2-D analogue?



  - ✓ Note: Just as with a face of the Rubik's Cube, the squares on the Rubik's Square are connected, i.e. Blue always shares a side with Green and Yellow but not with Red

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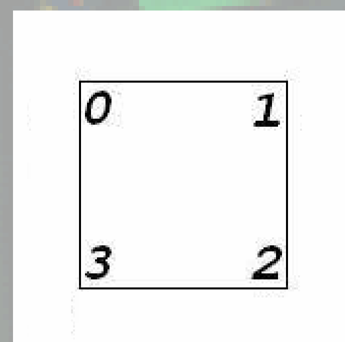
Representations

- Since working with the colors may be cumbersome, use two other representations:
  - ✓ Network graph:



  - ✓ Ordered quadruple: 0-1-2-3

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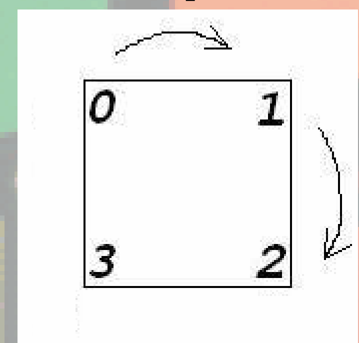
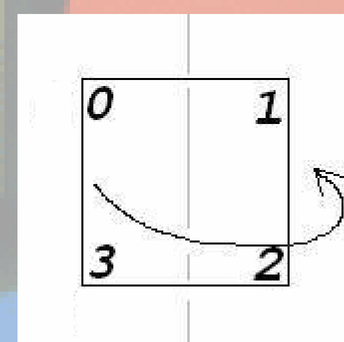
Moves

- What moves are legal to manipulate the Rubik's Square?
  - ✓ Rotate (clockwise); resulting quadruple 3-0-1-2; call this *R*



  - ✓ Flip (over *y*-axis); resulting quadruple 1-0-3-2; call this *F*



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Move Sequences

- Notice that other end states of the Rubik's Square can be generated by compositions of these two moves:
  - ✓ *FR* yields 2-1-0-3
  - ✓ *RF* yields 0-3-2-1
  - ✓ *RR* yields 2-3-0-1

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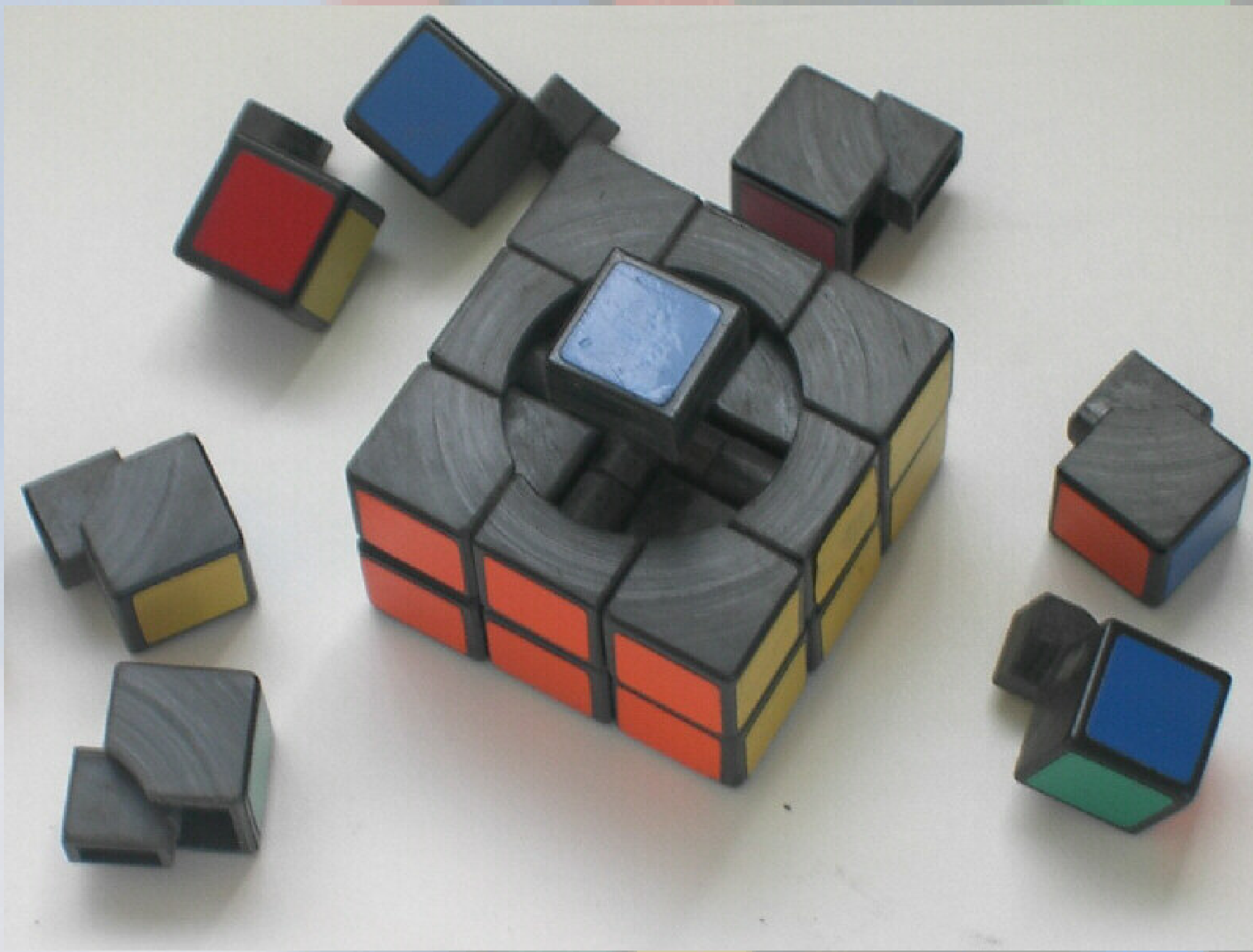
Move Sequences

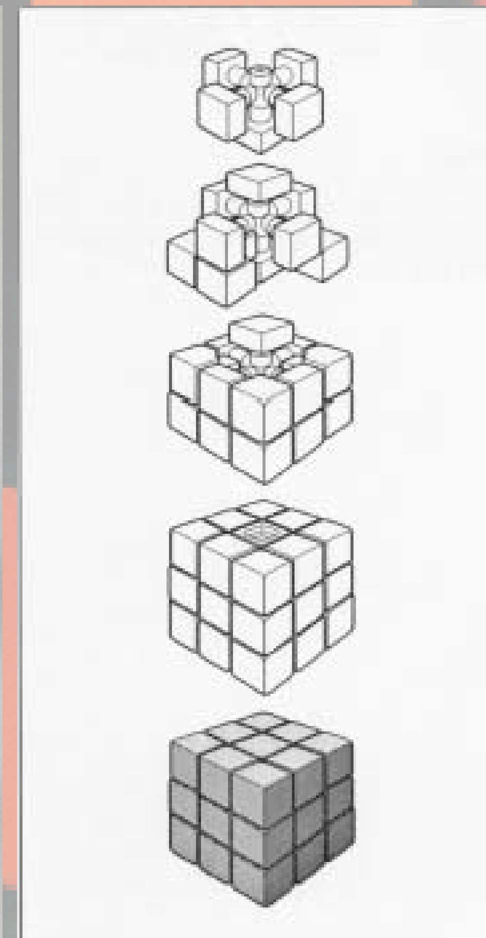
- Some move sequences yield the original state:
  - ✓ *FF* yields 0-1-2-3
  - ✓ *RRRR* yields 0-1-2-3
- Some move sequences undo other sequences:
  - ✓ *F* undoes itself
  - ✓ *RRR* undoes *R* (3 cw turns equals 1 ccw turn!)
- These are identity and inverses!

End States

- Can all possible end states be reached? Can all possible graphs and quadruples be reached?
- No… since some colors cannot be adjacent, a state like 0-2-1-3 is impossible (Blue adjacent to Red)
- Only 8 end states are possible, so “solving” the Square is easy

Rubik's… Cube!





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End states

- The geometry of the cube:
  - ✓ 6 center pieces with 1 orientation each
  - ✓ 12 edge pieces with 2 orientations each
  - ✓ 8 corner pieces with 3 orientations each
- $12! \cdot 2^{12} \cdot 8! \cdot 3^{8}$ permutations
- However, only 1/12th of the end states are reachable, so the number of permutations is lowered to 4.3×10^{18} quintillion

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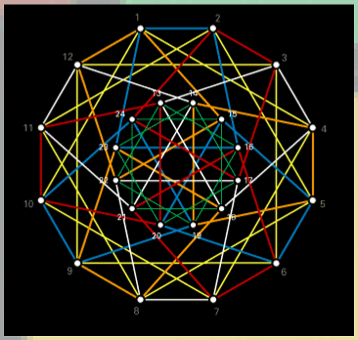
Representations

- The two other representations do not make the cube much easier:

  ✓ Network graph:



  ✓ Ordered list: 0-A-1-B-2-C-3-D-E-F-G-H-4-I-5-J-6-K-7-L (where A-L are the edges and 0-7 are the corners)

Group Theory

- A <u>group</u> is formed when a set has three qualities true with respect to an operation:
  - ✓ The operation is associative
  - ✓ The set contains an identity for that operation
  - ✓ Each element of the set has an inverse under that operation that is also within the set
- Is the Rubik's Cube a group with respect to its moves?

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Group Theory on the Cube

- Are moves associative?
  - ✓ Yes… m_1 followed by m_2 followed by m_3 can be viewed as either $(m_1 m_2) m_3$ or as $m_1(m_2 m_3)$
- Is there an identity move?
  - ✓ Yes… do nothing… or do the same move four times ($mmmm$)
- Does each move have an inverse?
  - ✓ Yes… Since each move consists of a 90° rotation clockwise, the inverse would be a 90° rotation counterclockwise

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Moves

- What are the moves in 3-D?
  - ✓ *F* = rotate front face
  - ✓ *B* = rotate back face
  - ✓ *L* = rotate left face
  - ✓ *R* = rotate right face
  - ✓ *U* = rotate up face
  - ✓ *D* = rotate down face

Jaap's Puzzle Page (www.geocities.com/jaapsch/puzzles/cubie.htm)

Moves

- Do we need any other moves?
- No, but it may be advantageous to move from this subgroup of moves to another group that is a superset to $\{F, B, L, R, U, D\}$ that has other useful elements.

Moves and Sequences

- Three examples
  - ✓ Easy: Just as with the Rubik's Square, *RRR* equals a counterclockwise turn, so call this *R*'. This yields six more moves.
  - ✓ Medium: A move of parallel faces, like *UD*', has the effect of rotating the middle "equator" slice around the cube's *y*-axis, so call this M_y. This yields six more moves.
  - ✓ Hard: A move like *FR* has the effect of moving the front upper edge to the right upper position. While this probably doesn't warrant a name, such maneuvers come in handy to solvers.

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