Candy Sharing Game: Lesson Plan

Students learn a candy sharing game with a simple rule for passing candy around the circle. Students will experiment to figure out how to make the game stop, end in a fixed point, or end in a cycle depending on the amount of starting candy, the number of people playing, and the initial distribution of candy.

Levels 4th through 12th grade

Topics Dynamical Systems, Number Patterns

Goals

- Students will learn what a dynamical system is.
- Students will learn the terms *fixed point* and *cycle*.
- Students will find an initial distribution of candy that causes the game to stop.
- Students will find an initial distribution of candy that leads to a fixed point.
- Students will find an initial distribution of candy that leads to a cycle.
- Students will consider the amount of candy they need to continue their game forever.
- Students will consider the largest amount of candy they can have in a game that leads to a cycle instead of a fixed point.

Preparation Time None once materials and copies are gathered

Activity Time 25 to 60 minutes for the introductory activity. It is possible to extend this into a sequence of classes relating to matrices or dynamical systems.

Materials and Preparation

- Wrapped candies (at least 4 for every student)
- One "Challenges" handout for each group of 3 to 10 students.
- One pencil or pen for each group of 3 to 10 students.
- An easel, blackboard, or whiteboard with markers or chalk for writing.

Sources

Boston Math Circle research page: http://www.themathcircle.org/researchproblems.htm
Pass the Candy – An Introduction to Recursive Equations by Maria Hernandez, NCSSM.

Background

Allow 5 minutes for the background discussion and for explaining the rules.

Tell the students that this activity will introduce them to an area of mathematics called *dynamical systems*. A dynamical system consists of a space and a rule. In the candy sharing game, the "space" consists of people sitting in a circle. The "rule" describes how people in the circle pass candy.

Dynamical systems have many applications – anything that grows or moves according to a repeating rule can be described by a dynamical system. Mathematicians, scientists, and engineers use dynamical systems to study traffic patterns, the way neurons in the brain exchange electrical signals, and the way water flows.

Explaining the Rules

Have the students sit in a circle on the ground or around a table. A group of 3 to 10 people works best. If you have more students, you may want to break them up into more than one circle to play the game, but you can explain the rules with everyone sitting together.

Explain how the game will work.

- Each circle will have a leader who starts the game by distributing the candy among participants. The distribution of candy will not necessarily be even – some students will start out with more and some students may start out with none. No eating candy during the activity! Each student will choose a piece of candy when we are done.
- During the game, students should keep their candy on the ground in front of them. They should make sure that the candy does not get lost under their legs and they should not hold it in their hands.
- When the leader says "Share!", everyone who has two or more pieces of candy in front of them gives one piece to the person on their right and one piece to the person on their left. Show them that they should use both arms to do this at the same time. People with one or zero pieces of candy do nothing.
- After the appropriate people have shared candy, the leader will say "Share!" again. This process repeats until the group sees a pattern emerge in the game.
- Several things might happen with the game.
    1. The game might stop because no one is passing candy any more.
    2. The game might settle down so that even though everyone passes candy every time, the amount of candy that each person has is always the same. This is called a *fixed point*.
    3. A repeating pattern might emerge in the way that the candy is shared. This is called a *cycle*.

Play a practice round all together just to learn the rules. For the practice game, use more than twice as much candy as there are people in the circle. Distribute the candy unevenly so that some students get several pieces and some get none.

During the practice round, ask everyone who will be sharing to raise her hand before you actually ask the group to share. Tell the students that they need to share carefully so that one piece goes to the pile of the person on their left and one piece goes to the pile of the person on their right. The game will not work if they throw the candy somewhere else. Check that everyone who has two or more pieces shares and that those who have one or no pieces do not share.

Prepare to break the students up into their groups and appoint a leader for each group. If you have enough adults, the leaders can be adults. Otherwise, appoint a student as the leader for each group. Each leader should get a *Challenges* handout and a pencil for recording strategies. Each group should try to figure out as many of the challenges as possible.

Allow groups to work on the challenges for 10 to 30 minutes depending on their age, their attention span, and the amount of time you want to spend on the activity.

Helpful Hints for the Concluding Discussion

Allow about 10 minutes for the concluding discussion. Once the smaller groups are finished, gather again as a larger group to discuss the results.

Here are some hints about answers to the challenge questions. It is not important that the students answer all of the questions. If they did not get very far with a particular question, you may wish to skip over it during the discussion. Focus on what they were able to figure out instead. During the large group discussion, you might try some of the distributions suggested below to illustrate different game outcomes.

- Find an initial distribution of candy that eventually causes the game to stop.

  The general strategy for this challenge is to use a small amount of candy compared with the number of people in the circle. If there are fewer pieces of candy than students, then the game will always stop. If the number of pieces of candy is equal to the number of students, the game may stop or it may continue depending on the distribution. If you give one of the students at least 2 pieces, then there will be at least one sharing motion before the game stops.

- Find an initial distribution of candy that leads to a *fixed point*.

  The general strategy for this challenge is to use a lot of candy compared with the number of people in the circle. If there are more than twice as many pieces of candy as students, then the game will always end in a fixed point. If there is exactly twice as much candy as people in the circle, then the game may go to a fixed point or a cycle depending on the distribution.

- Find an initial distribution of candy that leads to a *cycle*.

  The general strategy is to use an amount of candy that is between the number of people in the circle and twice that number.

- What is the smallest amount of candy that you can use to design a game that never stops?

  The smallest amount of candy is equal to the number of people in the circle. If there is less candy than people, then the game always comes to a stop once the candy spreads out. If you use the same amount of candy as people, then it is possible to distribute the pieces so that the game never stops. For example, give one student 2 pieces of candy, give one of the students next to her 0 pieces of candy, and give everyone else 1 piece of candy. This game will never stop and a pattern of sharing emerges.

- What is the smallest amount of candy that you can use if you want to guarantee that the game will never stop no matter how the leader distributes the candy? (This question is slightly different from the previous question because in that question you were allowed to tell the leader how to distribute the candy.)

The smallest amount of candy for this question is one more than the number of people in the circle. If the amount of candy is the same as the number of people, the game may stop if the distribution is not right. For example, if everyone in the circle gets 1 piece of candy, no one ever shares. If there is at least one more piece of candy than there are people, then the game does not stop.

- What is the largest amount of candy that you can use if you want a game that leads to a cycle instead of a fixed point?

  The largest amount of candy you can use if you want a cycle is equal to twice the number of people in the circle. If you use one less than this amount, then there will be a cycle no matter how the candy is distributed. If there are exactly twice as many pieces of candy as there are people, then the starting distribution needs to be carefully chosen. One way to make a cycle in this case is to give one student 3 pieces of candy, give one of the students next to her 1 piece of candy, and give all the other students 2 pieces of candy.

- What is the smallest amount of candy that you can use if you want to design a game that leads to a fixed point?

  The answer to this question is the same as the previous one. The smallest amount of candy is equal to twice the number of people in a circle. If you use one more than this amount, then the game will find a fixed point no matter what the starting distribution is. If you use exactly twice as much candy as there are students in the circle, then one way to cause a fixed point is to give each student 2 pieces of candy.

- What is the smallest amount of candy that you can use if you want to guarantee that the game will lead to a fixed point no matter how the leader distributes the candy?

  The answer is that you need one more than twice as many pieces of candy as people in the circle.

Last Words

Remind the students that this game is an example of a dynamical system. Mathematicians who study dynamical systems think about patterns of movement that emerge depending on the initial conditions.

If the students made it through all of the questions and still have some focus left, use an easel, whiteboard, or chalkboard to write the summary information below.

This is how a mathematician would summarize our discussion. Let C stand for the number of pieces of candy and let P stand for the number of people in the circle.

- If $C < P$, then the game always stops no matter what distribution is used.
- If $C = P$, then the game might stop or it might lead to a cycle depending on the distribution.
- If $P < C < 2P$, then the game always leads to a cycle.
- If $C = 2P$, then the game might lead to a cycle or it might lead to a fixed point.
- If $C > 2P$, then the game always leads to a fixed point.

There are several questions relating to this game that no one has figured out yet. The cases where $C = P$ or $C = 2P$ are not very well understood. If $C = P$, how can we tell what initial distributions will lead to a cycle and which ones will cause the game to stop? If $C = 2P$, how can we tell what distributions lead to a cycle and which ones lead to a fixed point? These are the sorts of questions that mathematicians ask when they think about dynamical systems.

Taking it Farther – Changing the Rules

Challenge students to experiment with changing the rules of the game. It can be helpful to code some of these games into a spreadsheet.

One variation is for each player to pass half their candy to the person on their left at each iteration. (If the amount is odd, players should keep one more than they pass.) For this game, there are three possible eventual outcomes. The first possible outcome is that everyone ends up with the same number of candies. The second outcome is that each person ends up with the same amount at each step, but the amounts are not equal. The third outcome is that the system never settles down completely, but extra pieces of candy move around the circle.

In this game is it possible to predict the eventual equilibrium given the starting conditions?

How does this rule change if players pass one more than they keep? What if players throw out one candy before dividing whenever they have an odd number? What if they take an additional candy before dividing whenever they have an odd number?

Another variation to explore requires players to keep one candy if they have an odd number and to keep two candies if they have an even number. They then give half of the rest of their candy to the person on their left and half to the person on their right.

Advanced students might like to use iterated matrices to find solutions to some of the questions posed above. Some of the questions can be modeled with matrix Markov Chains.

Students might like to create their own rules for passing candy and explore what happens for those games.