

Exploring Fractals through Geometry and Algebra

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Learning Objective and skills practiced

Students will:

Learn the three criteria for fractals

Understand mathematics as a developing science.

Develop the geometric relationship of self similar

Plot numbers in real and complex plane

Use complex numbers in graphing.

Develop an understanding of iterations.

Graph in the real plane lines, quadratic and cubic

Use triangular grid paper to start Sierpinski's Triangle

See a graphically representation of convergence, periodic and divergence.

Develop and use logarithm to "undo" exponentiation.

Use the software program excel

Use function notation

Square decimals

Materials: For each student team: copies of lab, ruler, calculator, graph paper, access to Excel or other spreadsheet software, triangular grid paper, and a marker. To make your own triangle graph paper go to <http://incompetech.com/graphpaper/triangle/>

Pre-Lab: Power point including an online construction of Sierpinski's Triangle. You may want students to explore the current and potential uses of fractals as a writing component to this activity.

Lab Activities:

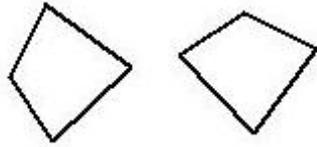
Each of these labs may be used by themselves or together to obtain a better understanding of fractals. If you wish to give these as worksheets for students we suggest you add appropriate spacing for answers.

Post-Lab: Further exploration Use the software program [fractint](#) a freeware program.

Deanna - math4fun101@yahoo.com Activity 1: Self-Similarity

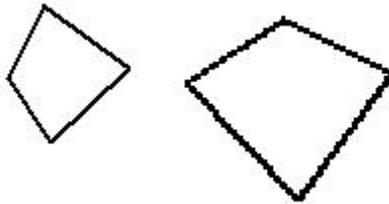
In this activity, we will be reviewing the geometric relationship of similarity and exploring a more specific relationship called self-similarity.

1. Consider these two shapes:



Are they the same? What is their geometric relationship?

2. Now, what about these two shapes?



Are they the same? What is their geometric relationship?

3. In your own words, describe what it means for two shapes to be congruent.
4. In your own words, describe what it means for two shapes to be similar.
5. Draw two rectangles that are not congruent but ARE similar.
6. Now, draw two rectangles that are NOT similar. Is it possible?
Explain in words why it is or is not possible.
7. Draw two squares that are not congruent but are similar.
8. Now, draw two squares that are NOT similar. Is it possible?
Explain in words why it is or isn't possible.

9. Is it possible to draw two congruent polygons that are also similar? Can you draw an example?

If two polygons are congruent, will they always be similar as well? Explain.

10. Is it possible to draw two similar polygons that are also congruent? Can you draw an example?

If two polygons are similar, will they always be congruent as well? Explain.

At this point, you have hopefully recalled that geometric shapes are *congruent* if they have exactly the same size and shape. Additionally, if two polygons are *similar*, then their corresponding angles are congruent and the lengths of the corresponding sides are proportional. Now we will continue to investigate similarity with the fractal named Sierpinski's Triangle.

11. In order to create Sierpinski's Triangle, we begin with an equilateral triangle and connect the midpoints of each side. Then, the middle is removed. This is what the first step looks like:



Do you notice any interesting patterns in the geometry? What fraction of the total area is the area of the triangle at the top? (Recall that the total area does NOT include the middle region)

12. The next step in creating our triangle is to take each of the new triangles, connect their midpoints, and remove the middle section.



Consider the top $\frac{1}{3}$ of the triangle.

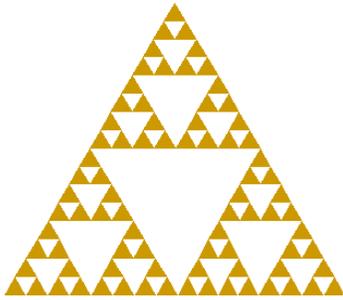
Draw just that portion here:

13. Compare your drawing to the whole Sierpinski's Triangle. In your words, how are the two shapes alike? How are they not alike?

14. What is the specific geometric relationship between these two triangles?

15. Is there another part of Sierpinski's Triangle that is like your drawing? How many are there total? What is their geometric relationship to each other?

16. If we continue to construct Sierpinski's Triangle, it will soon look like this.

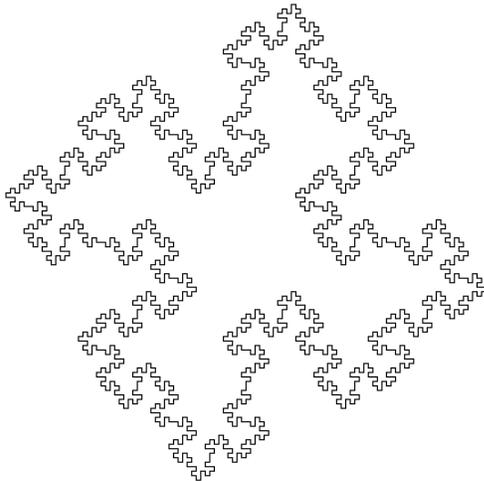


Do you notice any other similarity relationships appearing in the shape?

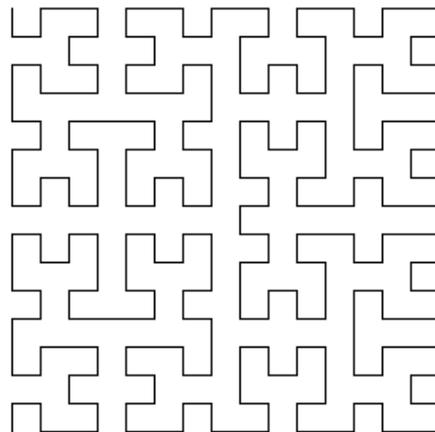
17. Compare the similarity of Sierpinski's Triangle to the similarity relationships of the polygons we investigated first? What is the difference between these similarities?

Sierpinski's Triangle is made up entirely of triangles that are similar to itself. We call this special relationship *self-similarity*. Some shapes are comprised completely of self-similarity while others can contain only some areas that are self-similar. Consider the following figures. Determine if they are self-similar, contain some self-similarity, or neither. Explain your answers.

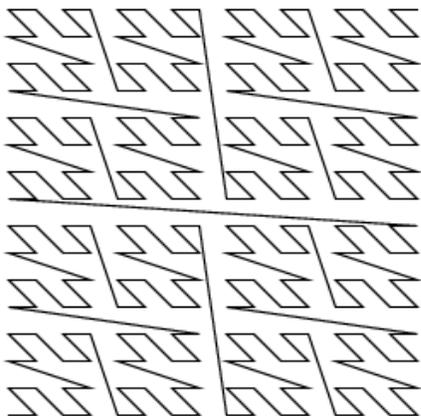
18.



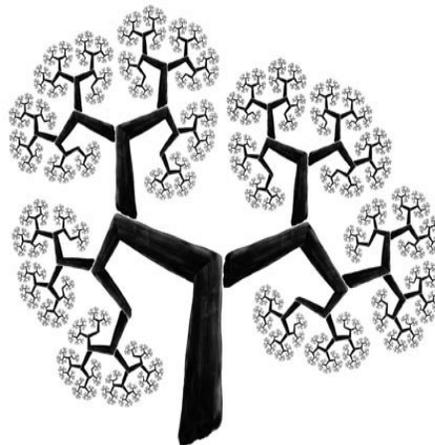
19.



20.



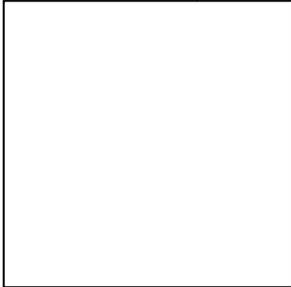
21.



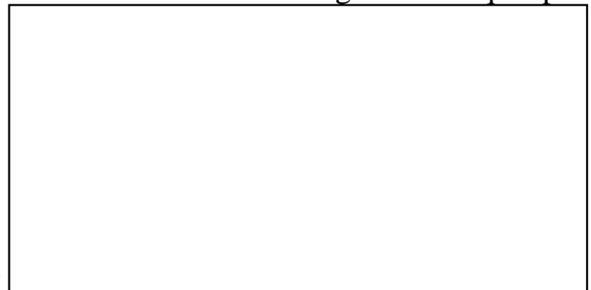
Laura – laurephant@hotmail.com Fractional dimensions an intro to logs - depends on students understanding self similarity

To understand the second component of fractals, fractional dimensions, we first need to examine what is meant by dimension.

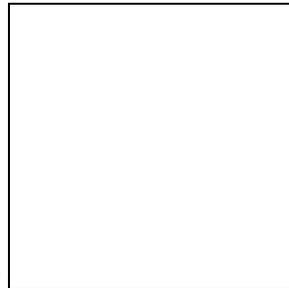
1. Can you draw lines through the square below which will cut the square into 4 equal pieces that look exactly like the original only smaller?



2. How many times would you have to multiply a little square side so that the area would be the same of the big square?
3. Can you draw lines through the rectangle below which will cut the rectangle into 4 equal pieces



- that look exactly like the original only smaller?
4. How many times would you have to multiply a little rectangle side so that the area would be the same of the big rectangle?
 5. Can you draw lines through the square below which will cut the square into 9 equal pieces that



- look exactly like the original only smaller?
6. How many times would you have to multiply a little square side so that the area would be the same of the big square?

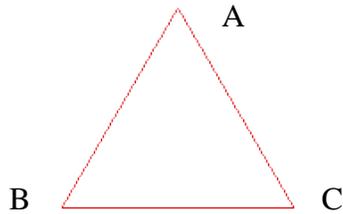
7. Can you draw a line through the object below which will cut the object into 2 equal pieces that

look exactly like the original only smaller?

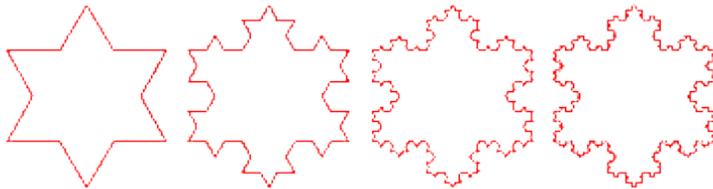
8. How many times would you have to multiply the little piece so that the length would be the same of the original?

19. First we need a new “undo” notation. What is the sign for addition? How you undo addition? What is the sign?
20. What is the sign for multiplication? How do you undo multiplication? What sign?
21. Right the math fact and the reverse math fact for the following: four times 3 is 12
22. What is the sign for squaring? How do you undo squaring? What sign?
23. How would you write 2 raised to the fifth power, **called exponentiation**? What if you want to undo so you only know the power? Just guess.
24. You probably realized in 4 we need notion for this idea. We call this “undo” **log**. Really we call it logarithm but pretty much every one shortens it to log. Look on your scientific or graphing calculator do you see the **log** button? What is the button the 2nd function associated with it?
25. .
26. Write the expression ten raised to the power of 5 is 100,000. Guess how you think the reverse of this might look. Remember log “undoes” of raising and the *answer is the exponent* we raised it to?
27. Did you figure out $\log 100,000 = 5$? We did have to put in the ten because the “log” by itself tells me we are dealing with a 10 base - the number being raised.
28. Now lets look at the reverse fact of raising 2 to the power of 5 is 32. Where does the 5 go? Can you guess were the 32 goes? The 2 is the base this time, not 10, so we are going to have to show this using subscript.
29. You should have written $\log_2 32 = 5$. Let’s try a few more. Write the fact and the reverse math fact for the following: 3 raised to the power of 4 is 81. 5 raised to the power of 4 is 625.
30. Evaluate the log 100 (think ok no base here so must be base of 10. So what power do I raise 10 to to get 100?)
31. Evaluate $\log_2 64$.
32. In #13 rewrite 64 as a power of 8. Now find the $\log_2 8$. What do we need to do with the exponent of 8 to get the same answer we got in 13?
33. Since all logs are just another way to write an exponent, there are a couple of rules that go with logs. Using #14 what do you think the rule is for raising log to a power?
34. The rule formally stated looks like $\log_b a^n = n(\log_b a)$. Try this, rewrite the log base 4 of 16 squared.
35. We are almost there. Now lets apply this to our formula for dimensions $S^D = N$. We need D equal something. So what do you think we need to “undo?”
36. Okay if you said log you’ve got it. Log both sides and bring down the D so you are multiplying it.
37. Did you get $D \log S = \log N$? Remember how to undo multiplication?
38. Write the formula to find the dimension.

Try it with a few fractals. Some fractals we can't see the number of times it would take to get back to the original size. We can restate the rule $SD = N$, $N = \text{NUMBER}$ of exact same size pieces in any iteration replacing an original piece and $S =$ the number we need to multiply the new little pieces to get back to the original SIZE piece. Look at the initial value, step one, below Koch's Snowflake starts as a triangle.

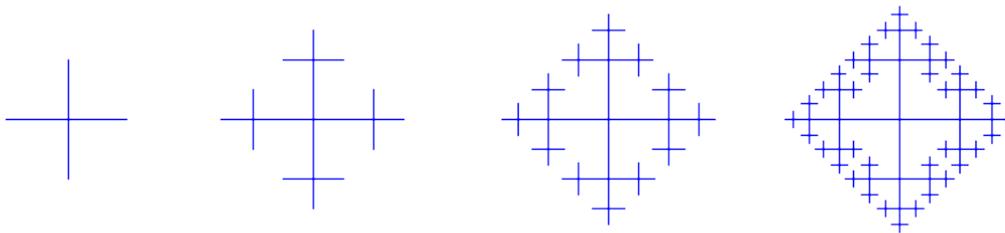


39. Draw the line segment AC in step one.
40. Look at step two below; draw what is in place of AC now.



41. Looking at AC how many pieces did we replace one side with?
42. Since all the pieces are the same size, how big is each compared with the original? So you would need to multiply the new little line segments by what?
43. Put it into the formula.
44. Try it with the next step.
45. Why is it the same decimal dimension even though we put in different numbers? What is the last rule we learned about fractals?
46. Think about: Can you find in the steps (iterations) of these fractals the dimensions?

Plusse fractal:



And for those of you who think this is only pictorial
<http://classes.yale.edu/fractals/Software/fracmusic.html>

Pat – pbparker@iusb.edu and Deanna math4fun101@yahoo.com Function, RANGE and DOMAIN and iteration

In this activity we investigate iterations and how they compare to other mathematical processes. In addition to this handout, you also need a triangular grid paper, a ruler, and a marker.

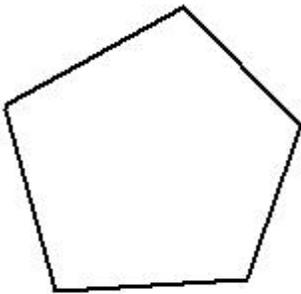
An **iteration** is, generally speaking, a mathematical rule that is repeated multiple times. Suppose you are given the following mathematical “rule.”

RULE: Connect the midpoints of all sides with line segments.

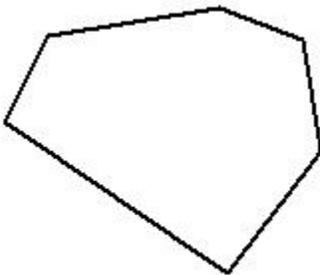
1. Apply the rule to the following shape.



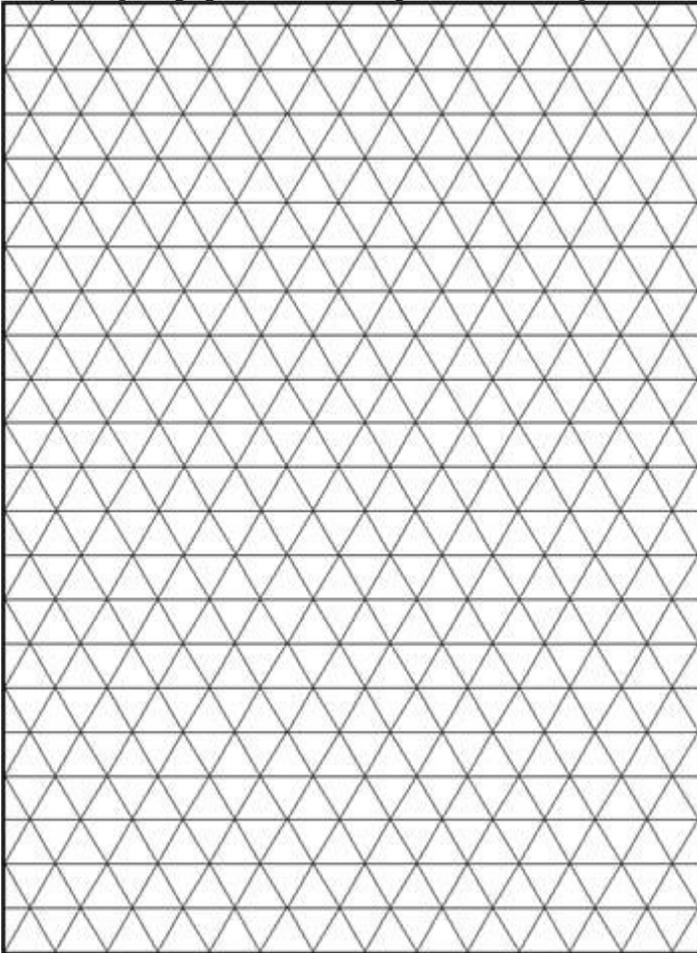
2. Now apply the rule to this shape.



3. Apply the rule again with this shape.



You have just applied a mathematical rule three times, yet you have NOT performed iterations. To get a better idea of how to iterate a mathematical rule, we will construct Sierpinski's Triangle by hand. On your grid paper, draw an equilateral triangle that is 12 units on all sides.



Informally, the “rule” used to create this triangle can be stated as **Sierpinski’s Rule**: Connect the midpoints of the sides with line segments and remove the region enclosed by those segments. We call our starting point an *initial value*.

4. Using your initial value, or initial triangle, apply the rule directly on your grid paper. Is there a component of the rule you may have difficulty applying with the available materials? You just applied the rule one time, and we call this result the **first iteration**.
5. Now, use the shape you have now and apply the rule again. (Remember that you have “removed” the middle section.)

You have now applied the rule twice, so your result is the **second iteration**.
6. Apply the **third iteration** on your shape. Hint: Your midpoints shouldn’t be on the grid this time, so a ruler will come in handy to make good line segments.
7. You just applied the mathematical rule three times. How does this process differ from the three times you applied the last rule?

8. You've only just begun to construct Sierpinski's Triangle. As with all fractals, the rule is iterated infinitely many times. How many iterations do you think you could perform on this grid paper?

One of the most common examples of mathematical rules is functions. Let's see how iterations are applied to functions.

9. In your own words, describe a function and give an example.
10. You may recall that a function is the relationship between x & y . This relationship, or sometimes called correspondence, is called a **FUNCTION**.
11. Build a table using this equation $x+4 = y$ to determine "x" inputs and "y" outputs.

Fill in the blanks.

x	y
1	5
2	6
3	7
4	8
5	9
6	..
7	11
8	..
..	..

Fig. 1

12. Since the "y" usually depends on the number "x," we call "y" the dependent variable and "x" is called the independent variable. One is the **RANGE** and the other the **DOMAIN**. Can you figure this out, if you know that "y" is depended on" x" ?
13. Suppose you had a linear function $f(x)=x+2$

The RULE is $(x+2)$. The INITIAL VALUE or SEED is $x=1$

Remember y is the dependent variable which corresponds to some number x (the independent variable). For example, if $y=f(x)$, the value of y that is determined by $x=5$ is denoted by $f(5)$.

Let's Practice

$$f(x) = (x + 2)$$

Each time the **RULE** is applied **using previous answer**, we call it an **ITERATION**.

$$f(1) = 1 + 2 = 3$$

$$f(3) = 3 + 2 = 5$$

$$f(5) = 5 + 2 = 7$$

$$f(7) = 7 + 2 = 9$$

Complete the next two iterations of this function.

$$f(\quad)$$

$$f(\quad)$$

14. Let's look at the next function $g(x) = (x^2 + 1)$

Perform 5 iterations using an initial value of $x=0$

The results of your last iteration should be 26.

15. What is the set of input values (x) called?

16. What is the set of output values (y) called?

17. How many y -values are assigned to each x value?

18. Do you think the graph of $y = x + 4$ and $F(x) = x + 2$ will have the same shape?

Testing for Convergence, Periodic, and Convergence with Iterations

1. Now that we know how to apply functions to iterations, we are going to explore some examples where we are given a function rule and what the long term behavior will be for that function rule and a given initial point. Some fractals can be produced by using a function rule and determining the long term behavior of the initial points. Examples are the Julia Sets and the Mandelbrot Set.
2. There are three different types of long term behavior that could occur:
 - **Converges:** The iterations approach a single value or stay at a single value for a long period of time and never leave that value. According to the American Heritage Dictionary, converges is “the property or manner of approaching a limit, such as a point, line, function, or value”.
 - **Periodic:** The iterations keep going through the same cycle of values in the long term. According to the American Heritage Dictionary, periodic is “having or marked by repeated cycles”.
 - **Diverges:** Any iteration that does not converge or does not show periodic. According to the American Heritage Dictionary, diverges “fails to approach a limit”.

Note: There is another more advanced definition for periodic and diverges.
Periodic behavior occurs if there are subsequences of the iterations that will converge. Divergence occurs if there are no subsequences of the iterations that will converge.

3. The following are some examples of sequences that converge:
1, 1.5, 1.75, 1.875, 1.9375, 1.96875, 1.984375, 1.9921875, ...
-2, -0.6, -0.18, -0.054, -0.0162, -0.00486, -0.001458, ...
-1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, ...

Can you see what values the sequences are approaching?

4. The following are some examples of sequences that are periodic:
2, 6, 7, 2, 6, 7, 2, 6, 7, 2, 6, 7, 2, 6, 7, ...
15, 27, -8, -32, 5, -2, 9, -32, 5, -2, 9, -32, 5, -2, 9, ...
-1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, 1, -1, ...
Can you see the values that are being repeated in the long term?

5. The following are some examples of sequences that diverge:
2, 4, 8, 16, 32, 64, 128, 256, 512, ...
5, 8, 11, 14, 17, 20, 23, 26, 29, ...
-4, 7, 23, 5, 12, -9, 67, 100, 0.5, ...
Can you tell that these sequences are diverging?

6. Now we are going to test a function rule and see what initial points will cause the iterations to converge, diverge, or become periodic. The following is the function rule:

$$x_{n+1} = (x_n)^2 - 1$$

Fill in the table below and determine the long term behavior of the initial points.

x	x_1	x_2	x_3	x_4	Converges, Diverges, or Periodic	If Convergent, what is the iterations approaching
0.7						
-0.8						
1.3						
4						
-1.1						
-7						
-1						
1						
0						

7. A special type of point that commonly arises is a **fixed point**. This is a point that gets mapped to itself by the function rule. In the table above, which values were fixed points?

8. Based on the results in the table, can you see what interval of values for x will cause the iterations to converge, diverge, or become periodic?

9. Let's try the process again with a different function rule:

$$x_{n+1} = \frac{1}{x_n}$$

Fill in the table below and determine the long term behavior of the initial points.

x	x_1	x_2	x_3	x_4	Converges, Diverges, or Periodic	If Convergent, what is the iterations approaching
0.7						
-0.8						
1.3						
4						
-1.1						
-7						
-1						
1						
0						

10. Were there any fixed point iterations? Any points that produced periodic behavior?

11. Based on the results in the table, can you see what interval of values for x will cause the iterations to converge, diverge, or become periodic?

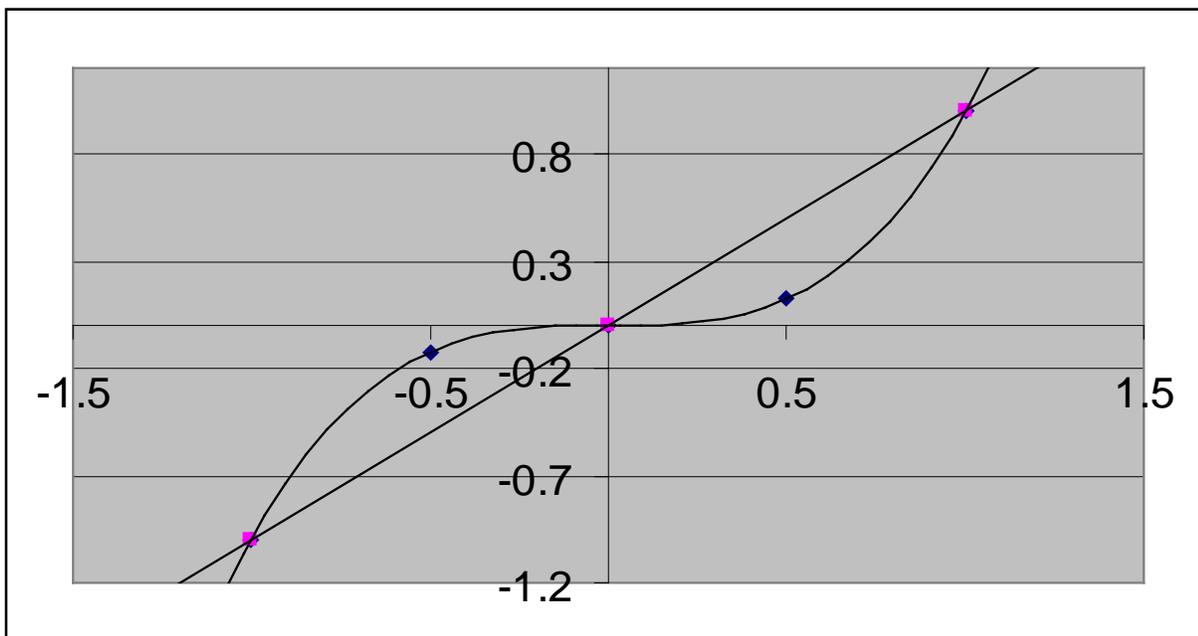
12. Another way to help in determining the long run behavior is using a **webbing process** on an xy graph. We are going to use this process on the function rule

$$x_{n+1} = (x_n)^2$$

There are two unknowns in the equation. Let $x_n = x$ and let $y = f(x) = f(x) = x_{n+1}$

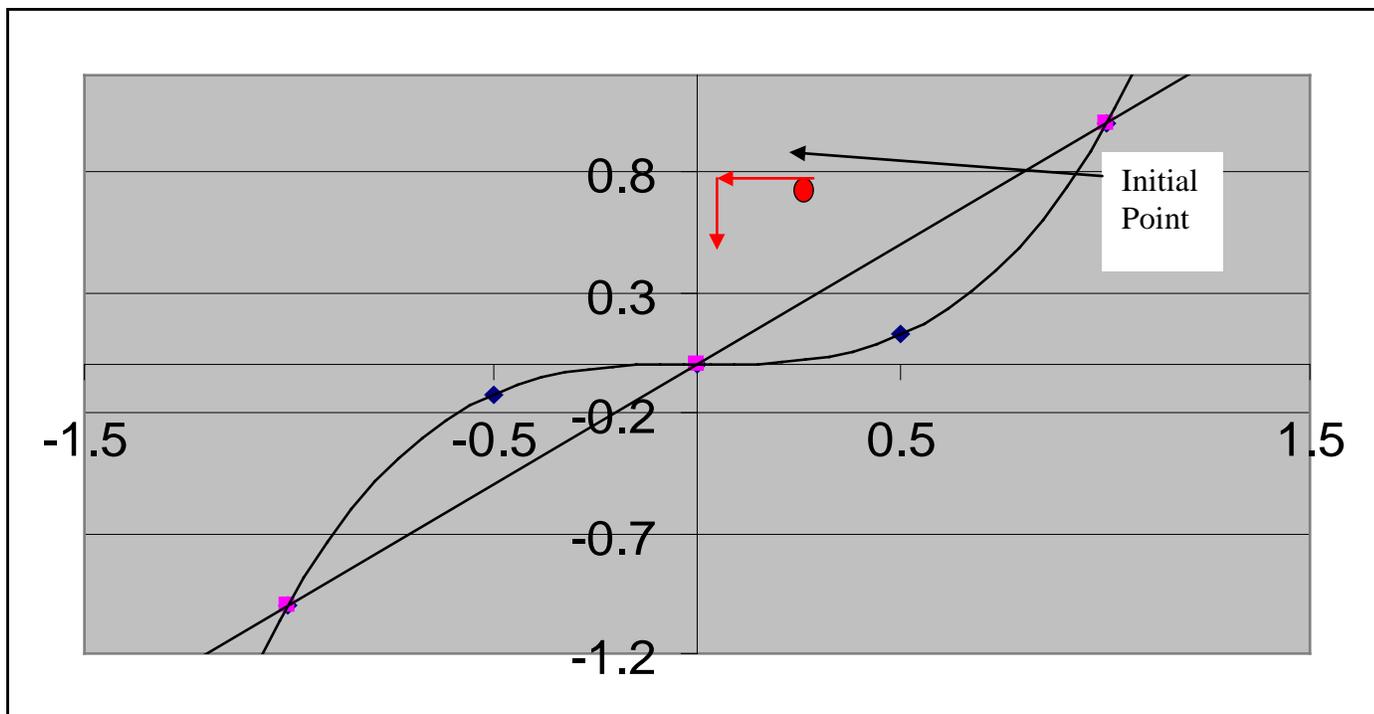
Therefore, the function rule becomes $y = x^2$

13. Remember on how to do an iteration? The output from the first step should become the input in the next step. In terms of x and y, the y value from one step should become the x value for the next step. Therefore, we are also going to graph the line $y = x$. This line will be used in the webbing process to convert the current y value to an x value. Below is a graph of $y = x$ and $y = x^2$ on the same graph.



14. We are going to test the initial point $x = 0.7$. At this value, on the $y = x^2$ curve, the y value is

$0.7\hat{z} = 0.343$. This will be our starting point. From this starting point, go directly across to the $y = x$ line. Then go down to the $y = x\hat{z}$. Continue this process. Can you tell what your web is approaching?



15. Can you see that the web approaches zero? Try some other initial values and see if you get the same results as before with your table on #6.

16. Try the cobwebbing process for the function $x_{n+1} = 2x_n$. What initial values converge, diverge, or have periodic behavior?
17. Try the cobwebbing process for the function $x_{n+1} = 0.7x_n$. What initial values converge, diverge, or have periodic behavior?
18. Now you try with $x_{n+1} = -x_n$

Kelly – kelly.deckelman@gmail.com Graphing in the complex plane by hand and using excel
 Here's my rough draft of my topic. I think we may be able to beef it up with more questions, but I'd like everyone's input. Because I have to walk kids through things there's much more instruction than question. We'll see Graphing in the complex plane is very much like graphing in the real number plane.

1. Draw coordinate axes as if you were going to plot a real point such as (3, 4). Instead of an x- and y-axis, we have a real- and imaginary-axes. The “old” x-axis is now the real-axis. Label the real-axis. The “old” y-axis is now the imaginary-axis. Label the imaginary-axis as well. You have just drawn the complex plane and are now ready to plot complex points.
2. Plot the point (3, 4) as if you were plotting on the real number plane (x- and y-axis.) Because we changed the names of what the axes represent, you have actually just plotted the complex number $3 + 4i$ in the complex number plane. The real part is 3, thus we go 3 units along the real number axis. The imaginary part is $4i$ thus we go 4 units along the imaginary axis.
3. Now that you can plot complex points in the complex plane, let's see how some fractals are created using the computer. Open Excel on your computer. Create 4 columns and label them A, B, Creal, and Cimag. A represents the real part of the complex number $Z = A + Bi$ and B represents the imaginary part. Creal and Cimag stand for the real and imaginary parts of the complex number C ($C = \text{Creal} + \text{Cimag}$).
4. A Julia Set is created by examining what happens to complex numbers when applying the function $Z_{n+1} = (Z_n)^2 + C$. If $Z_n = A + Bi$ and $C = \text{Creal} + i\text{Cimag}$, determine the equations for the real and imaginary parts of Z_{n+1} .
5. Let's start with a simple example to see if you have the correct equations. Enter $A=0$, $B=0$, $\text{Creal} = 0.32$, and $\text{Cimag} = 0.04$ into the first row of data in the excel spreadsheet (row 2). Now enter your equations for the real and imaginary parts of Z_{n+1} in the next A and B data cells (A3 and B3). To do this, start with an equal sign and refer to the existing cells A2, B2, C2, and D2. For example = $A2^2 + C2$. We will be copying this formula down the page to see what happens to A and B as we repeat the function $Z_{n+1} = (Z_n)^2 + C$. However, Creal and Cimag never change. So when entering the equations in excel refer to Creal as $\$C\2 and Cimag and $\$D\2 . This ensures that the values stay the same when we copy down.
6. If you have the correct equations A3 is 0.32 and B3 is .04. Copy your equations down the page by selecting the cells A2 and B2 (where you typed the equations). Drag your selection down the page about 100 cells and press CTRL + D. This copies the equation down the page. If your equations are correct you should see the following. If not, go back and fix your equations.

X	Y	Creal	Cimag
	0	0	0.32 0.04
	0.32	0.04	
	0.4208	0.0656	
	0.492769	0.095209	
	0.553757	0.133832	
	0.608736	0.188221	
	0.655132	0.269153	
	0.676754	0.392662	

Copy A and B further down the page about 1000 cells.

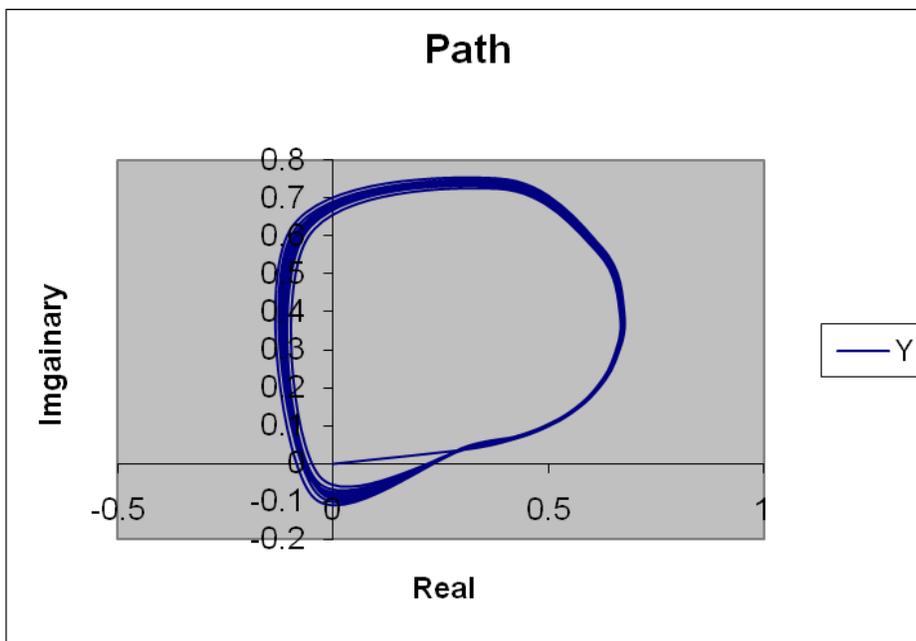
Experiment with different A and B starting values as well as different Creal and Cimag values.

7. What does it mean is a cell value says #NUM!?! (Hint: look at the numerical values in the cells preceding #NUM!?) What does it say about the original A and B if subsequent A and B values are #NUM!?! Do certain starting values of A and B produce #NUM! more quickly or more slowly than others? What does this mean?
8. Depending on the initial value of Z, we can see three types of behavior: convergent, divergent and periodic. What do you think it means to be periodic? (We'll explore this graphically later.)
9. Convergent means the list of outputs approaches a single point as the iterative function is applied to Z. In other words, starting at Z and infinitely iterating, you approach a single point. If an initial Z value produces #NUM! for subsequent cell values is it convergent? Why or why not?
10. For now we will say divergent means not convergent. If an initial Z value produces #NUM! for subsequent cell values is it convergent? Why or why not? What does #NUM! stand for again? How might you write a more specific definition of divergent at this point?
11. Are there initial Z values that produce neither convergent or divergent behavior?!

GRAPHING IN EXCEL

12. Click the Insert menu option followed by the Chart... option. Select XY (Scatter) as the chart type. Click the top right Chart sub-type for smooth connected lines without markers. Click the next button. Click the next button again. Enter titles for the axis labels. Select the gridlines tab and uncheck all options. Click the finish button. You will now see a graphical representation of all the changing A and B values (or the number Z).

For initial values $A = 0$, $B = 0$, $C_{real} = 0.32$, and $C_{imag} = 0.04$ the following graph is obtained.



13. What appears to be happening to the complex number $Z = 0 + 0i$ when we apply the iterative function $Z_{n+1} = (Z_n)^2 + C$ where $C = 0.32 + 0.04i$?
14. Is this behavior convergent, divergent or neither? What is the third type of behavior identified above? Do you think this behavior is periodic?
15. Write a definition of periodic behavior.
16. Is periodic behavior divergent or convergent? Justify your answer.
17. We will now define divergent behavior as neither convergent nor periodic.
18. Experiment with different initial Z values by changing $A2$ and $B2$. Can you find Z values that are convergent? Can you find Z values that are divergent? Can you find Z values that are periodic?
19. As a final thought, how do you think fractals get their color?

Candy Sharing Game: Leader Instructions

Students learn a candy sharing game with a simple rule for passing candy around the circle. Students will experiment to figure out how to make the game stop, end in a fixed point, or end in a cycle depending on the amount of starting candy, the number of people playing, and the initial distribution of candy.

Levels 4th through 12th grade

Topics Dynamical Systems, Number Patterns

Goals

- Students will learn what a dynamical system is.
- Students will learn the terms *fixed point* and *cycle*.
- Students will find an initial distribution of candy that causes the game to stop.
- Students will find an initial distribution of candy that leads to a fixed point.
- Students will find an initial distribution of candy that leads to a cycle.
- Students will consider the amount of candy they need to continue their game forever.
- Students will consider the largest amount of candy they can have in a game that leads to a cycle instead of a fixed point.

Preparation Time None once materials and copies are gathered

Activity Time 25 to 45 minutes

Materials and Preparation

- Wrapped candies (at least 4 for every student)
- One “Challenges” handout for each group of 3 to 10 students.
- One pencil or pen for each group of 3 to 10 students.
- An easel, blackboard, or whiteboard with markers or chalk for writing.

Primary Activity Source Boston Math Circle research page:

<http://www.themathcircle.org/researchproblems.htm>

Background

Allow 5 minutes for the background discussion and for explaining the rules.

Tell the students that this activity will introduce them to an area of mathematics called *dynamical systems*. A dynamical system consists of a space and a rule. In the candy sharing game, the “space” consists of people sitting in a circle. The “rule” describes how people in the circle pass candy.

Dynamical systems have many applications – anything that grows or moves according to a repeating rule can be described by a dynamical system. Mathematicians, scientists, and engineers use dynamical systems to study traffic patterns, the way neurons in the brain exchange electrical signals, and the way water flows.

Explaining the Rules

Have the students sit in a circle on the ground or around a table. A group of 3 to 10 people works best. If you have more students, you may want to break them up into more than one circle to play the game, but

you can explain the rules with everyone sitting together.

Explain how the game will work.

- Each circle will have a leader who starts the game by distributing the candy among participants. The distribution of candy will not necessarily be even – some students will start out with more and some students may start out with none. No eating candy during the activity! Each student will choose a piece of candy when we are done.
- During the game, students should keep their candy on the ground in front of them. They should make sure that the candy does not get lost under their legs and they should not hold it in their hands.
- When the leader says “Share!”, everyone who has two or more pieces of candy in front of them gives one piece to the person on their right and one piece to the person on their left. Show them that they should use both arms to do this at the same time. People with one or zero pieces of candy do nothing.
- After the appropriate people have shared candy, the leader will say “Share!” again. This process repeats until the group sees a pattern emerge in the game.
- Several things might happen with the game.
 1. The game might stop because no one is passing candy any more.
 2. The game might settle down so that even though everyone passes candy every time, the amount of candy that each person has is always the same. This is called a *fixed point*.
 3. A repeating pattern might emerge in the way that the candy is shared. This is called a *cycle*.

Play a practice round all together just to learn the rules. For the practice game, use more than twice as much candy as there are people in the circle. Distribute the candy unevenly so that some students get several pieces and some get none.

During the practice round, ask everyone who will be sharing to raise her hand before you actually ask the group to share. Tell the students that they need to share carefully so that one piece goes to the pile of the person on their left and one piece goes to the pile of the person on their right. The game will not work if they throw the candy somewhere else. Check that everyone who has two or more pieces shares and that those who have one or no pieces do not share.

Prepare to break the students up into their groups and appoint a leader for each group. If you have enough adults, the leaders can be adults. Otherwise, appoint a student as the leader for each group. Each leader should get a *Challenges* handout and a pencil for recording strategies. Each group should try to figure out as many of the challenges as possible.

Allow groups to work on the challenges for 10 to 30 minutes depending on their age, their attention span, and the amount of time you want to spend on the activity.

Helpful Hints for the Concluding Discussion

Allow about 10 minutes for the concluding discussion. Once the smaller groups are finished, gather again as a larger group to discuss the results.

Here are some hints about answers to the challenge questions. It is not important that the students answer all of the questions. If they did not get very far with a particular question, you may wish to

skip over it during the discussion. Focus on what they were able to figure out instead. During the large group discussion, you might try some of the distributions suggested below to illustrate different game outcomes.

- Find an initial distribution of candy that eventually causes the game to stop.

The general strategy for this challenge is to use a small amount of candy compared with the number of people in the circle. If there are fewer pieces of candy than students, then the game will always stop. If the number of pieces of candy is equal to the number of students, the game may stop or it may continue depending on the distribution. If you give one of the students at least 2 pieces, then there will be at least one sharing motion before the game stops.

- Find an initial distribution of candy that leads to a *fixed point*.

The general strategy for this challenge is to use a lot of candy compared with the number of people in the circle. If there are more than twice as many pieces of candy as students, then the game will always end in a fixed point. If there is exactly twice as much candy as people in the circle, then the game may go to a fixed point or a cycle depending on the distribution.

- Find an initial distribution of candy that leads to a *cycle*.

The general strategy is to use an amount of candy that is between the number of people in the circle and twice that number.

- What is the smallest amount of candy that you can use to design a game that never stops?

The smallest amount of candy is equal to the number of people in the circle. If there is less candy than people, then the game always comes to a stop once the candy spreads out. If you use the same amount of candy as people, then it is possible to distribute the pieces so that the game never stops. For example, give one student 2 pieces of candy, give one of the students next to her 0 pieces of candy, and give everyone else 1 piece of candy. This game will never stop and a pattern of sharing emerges.

- What is the smallest amount of candy that you can use if you want to guarantee that the game will never stop no matter how the leader distributes the candy? (This question is slightly different from the previous question because in that question you were allowed to tell the leader how to distribute the candy.)

The smallest amount of candy for this question is one more than the number of people in the circle. If the amount of candy is the same as the number of people, the game may stop if the distribution is not right. For example, if everyone in the circle gets 1 piece of candy, no one ever shares. If there is at least one more piece of candy than there are people, then the game does not stop.

- What is the largest amount of candy that you can use if you want a game that leads to a cycle instead of a fixed point?

The largest amount of candy you can use if you want a cycle is equal to twice the number of people in the circle. If you use one less than this amount, then there will be a cycle no matter how the candy is distributed. If there are exactly twice as many pieces of candy as there are people, then the starting distribution needs to be carefully chosen. One way to make a cycle in this case is to give one student 3 pieces of candy, give one of the students next to her 1 piece of candy, and give all the other students 2 pieces of candy.